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Determination of Heliocentric Elliptic Orbit

M. A. Rahman*
Missiles & Space Division, Douglas Aircraft Co., Santa Monica, Calif.
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In this paper a method of determining heliocentric elliptic orbits is presented. The procedure makes use of the following assumptions.

Symbols

ϕ	= a differentiable scalar field
∇	$= \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$
$\mathbf{r}_{1-2}$	= radius vector from heliocenter to perihelion
$\mathbf{r}_{1-3}$	= radius vector from heliocenter to a point on orbit
$\hat{\mathbf{P}}, \hat{\mathbf{Q}}, \hat{\mathbf{W}}$	= unit vectors along x_w, y_w, z_w axes of orbit plane, respectively
e	= eccentricity
$\hat{\mathbf{x}}$	= unit vector along x-axis
β	= angle between orbit plane radius vector and major axis of ellipse
x_w, y_w	= x and y axes of orbit plane
a, b	= major and minor axes of elliptic orbit
r_i	= $\lvert\mathbf{r}_{1-2}\rvert$ and $\lvert\mathbf{r}_{1-3}\rvert$
E	= eccentric anomaly
M	= mean anomaly

(1) The coordinates (x_1, y_1, z_1) of position of the radius vector $\vec{\mathbf{R}}_1$ from geocenter to heliocenter are known.

(2) The coordinates (x_2, y_2, z_2) of position of the radius vector $\vec{\mathbf{R}}_2$ from geocenter to perihelion are known.

(3) The coordinates (x_3, y_3, z_3) of position of the radius vector $\vec{\mathbf{R}}_3$ from geocenter to any point on the orbit of the planet are known.

Knowing the positions of these three radius vectors, we take the following steps out to find the various orbital elements.

(1) The angle or slope that the major orbital axis makes with x-axis of $z = 0$ plane is determined as follows:

The points of projections of (x_1, y_1, z_1) and (x_2, y_2, z_2) are $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$, respectively.

The slope m_{1-2} of a line through $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ is

$$m_{1-2} = (y_2 - y_1)/(x_2 - x_1) \tag{1}$$

Similarly, the slope m_{1-3} of a line through $(x_1, y_1, 0)$ and $(x_3, y_3, 0)$ is

$$m_{1-3} = (y_3 - y_1)/(x_3 - x_1) \tag{1a}$$

(2) The equation of the orbit plane through (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is found by the familiar use of

$$AX + BY + CZ + D = 0 \tag{2}$$

Substituting the coordinates of these points into Eq. (2), the values of the constants can be determined and, hence, the equation of the plane can be obtained. From this, one can write

$$\phi(x, y, z) = C \tag{3}$$

Again, the equation of the plane at the geocenter is

$$z = 0 \tag{4}$$

By transferring Eq. (4) to the heliocenter, one can write for the plane

$$\phi(z) = C \tag{5}$$

* Associate Engineer.

When the origin is transferred to the heliocenter, the axes remaining parallel to their original positions, the coordinates of the perihelion become $(x_2 - x_1)$, $(y_2 - y_1)$, and $(z_2 - z_1)$, and thus

$$|\mathbf{r}_{1-2}| = q = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{6}$$

and the length of the projection of $\mathbf{r}_{1-2}$ on $z = C$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{7}$$

Similarly, for point (x_3, y_3, z_3),

$$|\mathbf{r}_{1-3}| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2} \tag{8}$$

(3) The vector perpendicular to orbit plane at the heliocenter, from Eq. (3), is given by

$$\nabla(\phi_1) = \nabla\phi(x, y, z) \tag{9}$$

Thus, the unit vector

$$\hat{\mathrm{w}} = \frac{\nabla\phi(x, y, z)}{|\nabla\phi_1|} \tag{10}$$

and the vector perpendicular to the plane $z = C$ at the heliocenter, from Eq. (5), is given by

$$\nabla(\phi_2) = \nabla\phi(z) \tag{11}$$

The angle between $\nabla(\phi_1)$ and $\nabla(\phi_2)$ is

$$\theta = \cos^{-1}\frac{\nabla(\phi_1)\cdot\nabla(\phi_2)}{|\nabla(\phi_1)|\,|\nabla(\phi_2)|} \tag{12}$$

This is also the angle of inclination.

After bringing the x-axis in line with Eq. (7) by using Eq. (1), the vector through the node is

$$\nabla(\phi_1)x\nabla(\phi_2)$$

The unit vector through the node

$$\hat{\mathbf{N}} = \frac{\nabla(\phi_1)x\nabla(\phi_2)}{|\nabla(\phi_1)x\nabla(\phi_2)|} \tag{13}$$

The angle between the node and the x-axis is

$$\psi = \cos^{-1}\frac{\hat{\mathbf{N}}\cdot\hat{\mathbf{x}}}{|\hat{\mathbf{N}}|\,|\hat{\mathbf{x}}|} \tag{14}$$

$$\hat{\mathbf{x}} = (\hat{\mathbf{x}}\cdot\hat{\mathbf{P}})\hat{\mathbf{P}} = \hat{\mathbf{P}}\cos\alpha$$

therefore,

$$\hat{\mathbf{P}} = \hat{\mathbf{x}}/\cos\alpha \tag{15}$$

where α is the angle between Eqs. (6) and (7). The unit vector

$$\hat{\mathbf{Q}} = \hat{\mathbf{P}}x\hat{\mathbf{W}} \tag{16}$$

From Eq. (6) and conic relationships,[4] the semilatus rectum is

$$p = 2q = 2|\mathbf{r}_{1-2}| = b^2/a \tag{17}$$

and

$$e = p - r_i/x_{wi} \tag{18}$$

Using Eqs. (1), (1a), (6), and (8),

$$x_{wi} = r_i\cos\beta \tag{19}$$

$$a = q/(1 - e) \tag{20}$$

From Eq. (144) of Ref. 4,

$$\cos E_i = e + x_{wi}/a$$

and

$$\sin E_i = y_{wi}/a\sqrt{(1 - e^2)}$$

Here, $i = 1$ and 2.

From Kepler's equation [Eq. (4) of Ref. 4],

$$M_1 = E_1 - e\sin E_1 = n(t - T)_1, \quad 1 \rightarrow 2$$

and knowing[4]

$$n = k\sqrt{m_s + m_P}\,a^{-3/2}$$

where $k\sqrt{m_s + m_P}$ = constant, the time interval

$$\tau_{1-2} = (M_2 - M_1)a^{3/2} \tag{21}$$

Thus, using the above equations, all elements of the orbit can be obtained.

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A Supplementary Note on the Optimum Design of Box Beams for Combined Strength and Stiffness

B. Saelman

Design Specialist, Lockheed California Co., Division of Lockheed Aircraft Corp.

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Reference 1 contains a discussion of some optimum-design factors for a wing box beam in which strength and stiffness requirements must be met.

Prior to this stage of design, and particularly in the case of swept wings, one is aware that the actual loads on the structure are functions of its torsional and bending stiffnesses. It follows that a judicious distribution of material can minimize the external loads on the wing. We have

$$M = \int_s^l \int_s^l l\,ds\,ds = \int_s^l \int_s^l \frac{qc}{144}\,(C_{L\alpha}\alpha_g + C_{L\alpha_e}\alpha_s)\,ds\,ds \tag{1}$$

$$\alpha_s = \phi\cos\Lambda - \Gamma\sin\Lambda = \cos\Lambda \int_0^s \frac{T}{GJ}\,ds - \sin\Lambda \int_0^s \frac{M}{EI}\,ds \tag{2}$$

and

$$T = \int_s^l l e_1 c\,ds \tag{3}$$

It is seen from Eq. (1) that when α_s is a minimum, M is also a minimum. Since M and T are functions of α_s in the expression for α_s, it would be difficult to obtain directly a distribution of area so that α_s would be minimized at each station; and it is obvious that an iterative procedure is required in order that the final values of J, I, α_s, T, and M all be compatible in their relationships in the above equations. Furthermore, the distribution of material required for strength will not necessarily correspond to the distribution which yields minimum external loads.

It is seen from Eq. (2) that a high J and a low value of I would tend to minimize the incremental angle of attack under positive load conditions. We can write

$$\frac{GJ}{EI} = \frac{G}{E}\,\frac{4A^2/[(2c/t_s) + (2h/t_F)]}{(ct_e h^2)/2} \tag{4}$$

and for $t_s = t_e$, this ratio becomes

$$\frac{GJ}{EI} = \frac{4G}{E}\,\frac{1}{1 + (t_s/t_F)(h/c)} \tag{5}$$

This value is large when t_s/t_F and h/c are small.

Although it is difficult to distribute the material in the beam so that the bending moment is a minimum at *each* station, it is possible to distribute a fixed amount of material so that α_s is a minimum at the tip, or the material can be distributed so that α_s is a minimum at a specified station, at the same time keeping α_s at the tip at a fixed value.

$$\alpha_{sTIP} = \int_0^l \left[\frac{T(c+h)\cos\Lambda}{2GA^2t} - \frac{2M\sin\Lambda}{Ecth^2}\right] ds = \int_0^l \frac{Bds}{t} \tag{6}$$

and

$$W = \int_0^l \rho t p\,ds \tag{7}$$

By the calculus of variations we find that the optimum distribution of t for minimum α_s, based on a constant value of B, is given by

$$t = \frac{W}{\rho\sqrt{pA}}\;\frac{\sqrt{\dfrac{T(c+h)\cos\Lambda}{2G} - \dfrac{2Mc\sin\Lambda}{E}}}{\displaystyle\int_0^l \sqrt{\dfrac{T(c+h)\cos\Lambda}{2GA^2} - \dfrac{2M\sin\Lambda}{Ech^2}}\,\sqrt{p}\,ds} \tag{8}$$

The iterative process must, of course, be used to yield the distribution of t when B is adjusted to include the effects of loads due to deflection.

When α_s is to be a minimum at a given station and fixed at the tip, we have

$$\alpha_{s_1} + \alpha_{s_2} = \alpha_{sTIP} \quad \text{(fixed)} \tag{9}$$

and

$$W_1 + W_2 = W \quad \text{(fixed)} \tag{10}$$

Under the conditions given in Eqs. (9) and (10), α_{s_1} is to be minimum for the available W_1, while α_{s_2} is the allowable remaining deflection to meet the given tip deflection, α_{sTIP}, for a minimum W_2. Application of the Calculus of Variations yields two simultaneous equations in the Lagrange multipliers, each corresponding to one of the beam sections:

$$\left[\sqrt{p}\int_0^{s_1}\sqrt{B}\,\sqrt{p}\,ds\right]\sqrt{\lambda_1} + \left[\sqrt{p}\int_{s_1}^{l}\sqrt{B}\,\sqrt{p}\,ds\right]\sqrt{\lambda_2} = \alpha_{sTIP} \tag{11}$$

$$\left[\sqrt{p}\int_0^{s_1}\sqrt{B}\,\sqrt{p}\,ds\right]\frac{1}{\sqrt{\lambda_1}}\left[\sqrt{p} + \int_{s_1}^{l}\sqrt{B}\,\sqrt{p}\,ds\right]\frac{1}{\sqrt{\lambda_2}} = W \tag{12}$$

The solution of Eqs. (11) and (12) gives two values of λ_1, one of which is a minimum and the other a maximum, both compatible with the fixed tip deflection and total weight; at the same time the most efficient use of material in each segment is realized. Again, iteration must be used in the process in order to reflect the effects on the value of loads due to deflections, and the resulting quadratic equation must be solved for each iteration. The numerical values of the multipliers are used in the solution for the optimum distribution of material in each section. Ref. 2 discusses the use of the multiplier in a single-section beam.

Much attention has recently been given to the redundant analysis of internal-load distributions; however, it should also be noted that the stiffness distribution in a structure will affect the external loads.

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